Splitting Booleans with Normalization-by-Evaluation

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A Question from M. Shulman

Proof Assistants Beta

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Strong eta-rules for functions on sum types

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I am wondering whether a rule like the following is consistent with decidable conversion and type-checking for dependent type theory:

$$\frac{f\,g : (x : \mathsf{bool}) \to C\,x \qquad f\,\mathsf{tt} \equiv g\,\mathsf{tt} \qquad f\,\mathsf{ff} \equiv g\,\mathsf{ff}}{f \equiv g}$$

That is, if two functions with domain bool agree definitionally on tt and ff, then they are convertible. An analogous rule for functions on general inductive types like $\mathbb{N}$ is certainly

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Context: Martin-Löf Type Theories (with $\mathtt{Ty}, \Pi, \Sigma, x =_A y, \ldots$)

Extensional Principles in Intensional Type Theory

Type formers	Dec. of conv.	Reference
Functions $\Pi(x\colon A)B$	✓	[Coquand 96]
(Negative) records $\Sigma(x\colon A)B$	✓	[Norell 07]
Unit $\mathbb{1}$	✓	[Norell 07]
Identity $x =_A y$	×	[Castellan et al. 17]
Natural numbers $\mathbb{N}$	×	[RefNec]
Well-founded trees $\mathbb{W}(x\colon A)B$	×	Reduction to $\mathbb{N}$
Streams, $\mathbb{M}$-types	×	[McBride's Ripley]
Empty $\mathbb{0}$	×	[McBride's Ripley]
Booleans $\mathbb{B}$	???	

Booleans 101

Introductions

$$\frac{}{\Gamma \vdash \mathbb{B}}$$

$$\frac{}{\Gamma \vdash \texttt{tt} \colon \mathbb{B}} \qquad \frac{}{\Gamma \vdash \texttt{ff} \colon \mathbb{B}}$$

Dependent elimination

$$\frac{\begin{array}{cc}\Gamma \vdash b \colon \mathbb{B} & \Gamma, x \colon \mathbb{B} \vdash P \\ \Gamma \vdash t \colon P[\texttt{tt}/x] & \Gamma \vdash u \colon P[\texttt{ff}/x]\end{array}}{\Gamma \vdash \texttt{ind}(x.P;\, b;\, t \mid u) \colon P[b/x]}$$

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Computation rules

$$\Gamma \vdash \mathtt{ind}(x.P; \mathtt{tt}; t \mid u) \equiv t \ : \ P[\mathtt{tt}/x] \qquad \Gamma \vdash \mathtt{ind}(x.P; \mathtt{ff}; t \mid u) \equiv u \ : \ P[\mathtt{ff}/x]$$

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Extensionality (Naively)

$$\frac{\Gamma \vdash p \ : \ P[b/x] \qquad \Gamma, x \colon \mathbb{B} \vdash P \qquad \Gamma \vdash b \ : \ \mathbb{B}}{}$$

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Extensionality (Naively)

$$\frac{\Gamma \vdash p : P[b/x] \qquad \Gamma, x \colon \mathbb{B} \vdash P \qquad \Gamma \vdash b : \mathbb{B} \qquad \Gamma \vdash p[\texttt{tt}/b] \equiv t : P[\texttt{tt}/x] \qquad \Gamma \vdash p[\texttt{ff}/b] \equiv u : P[\texttt{ff}/x]}{\Gamma \vdash p \equiv \texttt{ind}(x.P; b; t \mid u) : P[b/x]}$$

Substitution is not enough: M. Baillon's counter-example

Assuming $\alpha : \mathbb{N} \to \mathbb{B} \in \Gamma$, consider

$$\Gamma \vdash \texttt{ind}(b.\forall n, \alpha\ n = b \to \mathbb{N};\ \alpha\ 42;\ \lambda\ n\ eq.\ 0 \mid \lambda\ n\ eq.\ 0)\ 42\ \texttt{refl} : \mathbb{N}$$

where $\quad \Gamma \vdash \texttt{refl} : \alpha\ 42 = \alpha\ 42$

Substitution is not enough: M. Baillon's counter-example

Assuming $\alpha : \mathbb{N} \to \mathbb{B} \in \Gamma$, consider

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$$\text{where} \qquad \Gamma \vdash \texttt{refl} : \alpha\ 42 = \alpha\ 42$$

But substituting α 42 by tt is ill-typed:

$$\Gamma \nvdash \texttt{ind}(b.\forall n, \alpha\ n = b \to \mathbb{N};\ \texttt{tt};\ \lambda\ n\ eq.\ 0 \mid \lambda\ n\ eq.\ 0)\ 42\ \texttt{refl} : \mathbb{N}$$

$$\text{where} \qquad \Gamma \nvdash \texttt{refl} : \alpha\ 42 = \texttt{tt}$$

Need to keep track of convertibility relations at $\mathbb{B}$!

Reflecting Conversion at $\mathbb{B}$

Add boolean constraints [ALTENKIRCH,DYBJER,HOFFMAN & SCOTT, 2001]

$$\frac{\Gamma \vdash \qquad \Gamma \vdash b \colon \mathbb{B} \qquad v \in \{\mathtt{tt}, \mathtt{ff}\}}{\Gamma, b \equiv v \vdash}$$

Reflecting Conversion at $\mathbb{B}$

Add boolean constraints [ALTENKIRCH,DYBJER,HOFFMAN & SCOTT, 2001]

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Extend conversion

$$\textsc{Reflection}\ \frac{(b \equiv v) \in \Gamma}{\Gamma \vdash b \equiv v \colon \mathbb{B}} \qquad \textsc{Explosion}\ \frac{(b \equiv \mathtt{tt}), (b \equiv \mathtt{ff}) \in \Gamma \qquad \Gamma \vdash t, u \colon C}{\Gamma \vdash t \equiv u \colon C}$$

$$\textsc{Cover}\ \frac{\Gamma \vdash b \colon \mathbb{B} \qquad \Gamma, b \equiv \mathtt{tt} \vdash t \equiv u \colon C \qquad \Gamma, b \equiv \mathtt{ff} \vdash t \equiv u \colon C}{\Gamma \vdash t \equiv u \colon C}$$

How to Check Conversion ?

Conversion checking may require full normal forms, e.g.

$$f : \mathbb{B} \to \mathbb{B} \vdash \qquad f \circ f \circ f \;\equiv\; f \qquad : \; \mathbb{B} \to \mathbb{B}$$

How to Check Conversion ?

Conversion checking may require full normal forms, e.g.

$$\begin{array}{rcccl} f\!:\mathbb{B}\to\mathbb{B}\vdash & f\circ f\circ f & \equiv & f & : & \mathbb{B}\to\mathbb{B} \\ f\!:\mathbb{B}\to\mathbb{B},\ x\!:\mathbb{B}\vdash & f\,(f\,(f\,x)) & \equiv & f\,x & : & \mathbb{B} \end{array}$$

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\end{array}
$$

$$
\begin{array}{rcccl}
f\!:\mathbb{B}\to\mathbb{B},\ x\!:\mathbb{B},\ x\equiv\mathtt{tt}\vdash & f\,(f\,(f\,\mathtt{tt})) & \equiv & f\,\mathtt{tt} & : \ \mathbb{B} \\
f\!:\mathbb{B}\to\mathbb{B},\ x\!:\mathbb{B},\ x\equiv\mathtt{ff}\vdash & f\,(f\,(f\,\mathtt{ff})) & \equiv & f\,\mathtt{ff} & : \ \mathbb{B}
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f : \mathbb{B} \to \mathbb{B},\ x : \mathbb{B},\ x \equiv \texttt{tt},\ f\,\texttt{tt} \equiv \texttt{tt} \vdash & f\,(f\,(f\,\texttt{tt})) & \equiv & \texttt{tt} & : \mathbb{B} \\
f : \mathbb{B} \to \mathbb{B},\ x : \mathbb{B},\ x \equiv \texttt{ff},\ f\,\texttt{ff} \equiv \texttt{tt} \vdash & f\,(f\,(f\,\texttt{ff})) & \equiv & \texttt{tt} & : \mathbb{B} \\
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$$
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\end{array}
$$

$$
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\end{array}
$$

For decidability, ultimately split on fully-normalized neutral booleans !

Evaluation

$$\mathtt{tt} \Downarrow \mathtt{tt} \qquad \mathtt{ff} \Downarrow \mathtt{ff}$$

$$\frac{b \Downarrow \mathtt{tt} \qquad t \Downarrow v}{\mathtt{ind}(x.P; b; t \mid u) \Downarrow v} \qquad \frac{b \Downarrow \mathtt{ff} \qquad u \Downarrow v}{\mathtt{ind}(x.P; b; t \mid u) \Downarrow v}$$

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$$\texttt{tt} \Downarrow \texttt{tt} \qquad \texttt{ff} \Downarrow \texttt{ff}$$

$$\frac{b \Downarrow \texttt{tt} \qquad t \Downarrow v}{\texttt{ind}(x.P; b; t \mid u) \Downarrow v} \qquad \frac{b \Downarrow \texttt{ff} \qquad u \Downarrow v}{\texttt{ind}(x.P; b; t \mid u) \Downarrow v}$$

$$\frac{b \Downarrow n \ \texttt{neutral}}{\texttt{ind}(x.P; b; t \mid u) \Downarrow \text{???}}$$

Evaluation

$$\texttt{tt} \Downarrow \texttt{tt} \qquad \texttt{ff} \Downarrow \texttt{ff}$$

$$\frac{b \Downarrow \texttt{tt} \qquad t \Downarrow v}{\texttt{ind}(x.P;\, b;\, t \mid u) \Downarrow v} \qquad \frac{b \Downarrow \texttt{ff} \qquad u \Downarrow v}{\texttt{ind}(x.P;\, b;\, t \mid u) \Downarrow v}$$

$$\frac{b \Downarrow n\ \texttt{neutral} \qquad t \Downarrow v \qquad u \Downarrow w}{\texttt{ind}(x.P;\, b;\, t \mid u) \Downarrow\ \texttt{bind}\ (\mathsf{case}\, v)\ (\lambda b.\mathtt{if}\, b\, \mathtt{then}\, v\, \mathtt{else}\, w)}$$

- [Abel & Sattler, 2019]: monadic evaluator to deal with boolean neutrals
- Evaluation reaches (weak) β-normal forms, reification η-expands

A domain for splitting booleans

Monadic evaluator eval : term × env → comp

```
type vl =
  | NfPi of { dom : vl ; cod : clos }
  | NfLam of clos
  | NfBool
  | NfU
  | NfNe of { ty : vl ; ne : nevl }
  | NfTrue
  | NfFalse

and comp = (nevl, vl) CT.case_tree

and nevl =
  | NeVar of int
  | NeApp of
    { fn : nevl ; arg : nfvl }

and nfvl =
  | Normal of { ty : vl ; tm : vl }

and clos = Clos of
  { env : env ; body : Tm.tm }

and env = comp list
```

Normalization procedure

Evaluation

$$[\![-]\!]_{-} : \texttt{semCtx} \times \texttt{term} \to \texttt{comp}$$

$$- \mathbin{\texttt{@}} - : \texttt{clos} \times \texttt{comp} \to \texttt{comp}$$

Reification

$$\lceil - \rceil_{-} : \texttt{semCtx} \times \mathcal{C}\,\texttt{vl} \to \mathcal{C}\,\texttt{nf}$$

$$[\![\lceil -]\!]_{-} : \texttt{semCtx} \times \texttt{comp} \to \mathcal{C}\,\texttt{vl}$$

The monad $\mathcal{C}$ splitting on boolean neutrals

```
type 'a t
val ret : 'a -> 'a t
val bind : 'a t -> ('a -> 'b t) -> 'b t
val case : ne -> bool t
val forall : bool Map.t -> (bool Map.t * 'a -> bool) -> 'a t -> bool
val equiv : bool Map.t -> 'a t -> 'a t -> bool
```

Normalization procedure

Evaluation

$$[\![-]\!]_{-} : \texttt{semCtx} \times \texttt{term} \to \texttt{comp}$$

$$- \mathbin{\texttt{@}} - : \texttt{clos} \times \texttt{comp} \to \texttt{comp}$$

Reification

$$\lceil - \rceil_{-} : \texttt{semCtx} \times \mathcal{C}\,\texttt{vl} \to \mathcal{C}\,\texttt{nf}$$

$$[\![\lceil - \rceil]\!]_{-} : \texttt{semCtx} \times \texttt{comp} \to \mathcal{C}\,\texttt{vl}$$

The monad $\mathcal{C}$ splitting on boolean neutrals

- Implemented with binary trees labelled by boolean neutrals
- Invariants: no duplicate case split, no redundant branches
- Renormalization either $\begin{cases} \text{at every bind} \\ \text{on observations } (\texttt{forall}, \texttt{equiv}) \end{cases}$

Algorithmic (Semantic) Typechecking

$\boxed{\Gamma \vdash A \lhd}$ $\boxed{\Gamma \vdash a \lhd A}$ $\boxed{\Gamma \vdash a \lhd_v V}$ $\boxed{\Gamma \vdash t \rhd A}$

Algorithmic (Semantic) Typechecking

$$\boxed{\Gamma \vdash A \lhd} \qquad \boxed{\Gamma \vdash a \lhd A} \qquad \boxed{\Gamma \vdash a \lhd_v V} \qquad \boxed{\Gamma \vdash t \rhd A}$$

$$\textsc{Head}\ \frac{\forall (\Xi \rightsquigarrow V) \in A, \quad \Gamma, \Xi \vdash t \lhd_v V}{\Gamma \vdash t \lhd A} \qquad \textsc{Conv}\ \frac{\Gamma \vdash t \rhd A \qquad \lceil A \rceil_\Gamma = \lceil \mathtt{ret}^{\mathcal{C}}\ B \rceil_\Gamma}{\Gamma \vdash t \lhd_v B}$$

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$$\textsc{If}\ \frac{\begin{array}{c}\Gamma \vdash b \lhd_v \mathbb{B} \\ \forall(\Xi \rightsquigarrow v) \in [\![b]\!]_\Gamma, \quad (v = \mathtt{tt} \implies \Gamma, \Xi \vdash t \rhd A) \wedge (v = \mathtt{ff} \implies \Gamma, \Xi \vdash u \rhd B)\end{array}}{\Gamma \vdash \mathtt{if}(b, t, u) \rhd \mathtt{let}^{\mathcal{C}}\, v = [\![[\![b]\!]_\Gamma]\!]_\Gamma\ \mathtt{in}\ \mathtt{if}\ v\ \mathtt{then}\ A\ \mathtt{else}\ B}$$

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$$\boxed{\Gamma \vdash A \lhd} \qquad \boxed{\Gamma \vdash a \lhd A} \qquad \boxed{\Gamma \vdash a \lhd_v V} \qquad \boxed{\Gamma \vdash t \rhd A}$$

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$$\textsc{App}\ \frac{\Gamma \vdash t \rhd T \qquad \forall(\Xi \rightsquigarrow V) \in T, \exists A, B,\ T = \Pi\, A\, B \qquad \Gamma \vdash u \lhd \mathcal{C}(\mathrm{dom}^{\Pi})\ T}{\Gamma \vdash t\ u \rhd \mathtt{let}^{\mathcal{C}}\ v = T\ \mathtt{in}\ [\![\mathrm{cod}^{\Pi}\ v\ @\ [\![u]\!]_\Gamma]\!]_\Gamma}$$

Conclusion

Current state:

- Ongoing work on a toy normalization-by-evaluation typechecker in ocaml
- 2 slightly different implementations of nbe
- 3 typecheckers
  - purely syntactic, conversion using NbE
  - semantic typechecking, shared global monadic context
  - semantic typechecking, monadic context local to each types

Future steps:

- A proof of normalization ? WIP formalization in Coq using [Bocquet, Kaposi & Sattler, 2023]
- Are the different implementation strategies equivalent ?
- Efficient/Reasonable implementations ? Maximal multi-focusing à la [Scherer, 2018] ?
- Correctness of the implemented typechecker ?

Checking types

$\boxed{\Gamma \vdash A \lhd}$ A is a well-formed type in the semantic context Γ

$$\textsc{Univ}\ \frac{}{\Gamma \vdash \mathtt{Ty} \lhd} \qquad \textsc{Bool}\ \frac{}{\Gamma \vdash \mathbb{B} \lhd} \qquad \textsc{Pi}\ \frac{\Gamma \vdash A \lhd \qquad \Gamma.[\![A]\!]_\Gamma \vdash B \lhd}{\Gamma \vdash \Pi\, A\, B \lhd}$$

$$\textsc{If}\ \frac{\begin{array}{c}\Gamma \vdash b \lhd \mathbb{B} \\ \forall (\Xi \rightsquigarrow v) \in [\![b]\!]_\Gamma, \quad (v = \mathtt{tt} \implies \Gamma, \Xi \vdash A \lhd) \wedge (v = \mathtt{ff} \implies \Gamma, \Xi \vdash B \lhd)\end{array}}{\Gamma \vdash \mathtt{if}(b, A, B) \lhd}$$

$$\textsc{Neut}\ \frac{\mathsf{ne}\ t \qquad \Gamma \vdash t \rhd A \qquad \forall (\Xi \rightsquigarrow V) \in A, V = \mathtt{Ty}}{\Gamma \vdash t \lhd}$$

Checking terms

$\boxed{\Gamma \vdash t \lhd A}$ t checks against the semantic type A in the semantic context Γ

$$\textsc{Head}\ \frac{\forall (\Xi \rightsquigarrow V) \in A, \quad \Gamma, \Xi \vdash t \lhd_v V}{\Gamma \vdash t \lhd A}$$

$\boxed{\Gamma \vdash t \lhd_v A}$ t checks against the semantic value type A in the semantic context Γ

$$\textsc{Conv}\ \frac{\Gamma \vdash t \rhd A \qquad \lceil A \rceil_\Gamma = \lceil \mathtt{ret}^{\mathcal{C}}\, B \rceil_\Gamma}{\Gamma \vdash t \lhd_v B} \qquad \textsc{PiU}\ \frac{\Gamma \vdash A \lhd_v \mathtt{Ty} \qquad \Gamma.[\![A]\!]_\Gamma \vdash B \lhd_v \mathtt{Ty}}{\Gamma \vdash \Pi\, A\, B \lhd_v \mathtt{Ty}}$$

$$\textsc{BoolU}\ \frac{}{\Gamma \vdash \mathbb{B} \lhd_v \mathtt{Ty}} \qquad \textsc{BoolCst}\ \frac{v \in \{\mathtt{tt}, \mathtt{ff}\}}{\Gamma \vdash v \lhd_v \mathbb{B}} \qquad \textsc{Lam}\ \frac{\Gamma.A \vdash t \lhd [\![B \mathbin{@} \mathtt{v}_0]\!]_{\Gamma.A}}{\Gamma \vdash \lambda\, t \lhd_v \Pi\, A\, B}$$

Inferring types for terms

$\boxed{\Gamma \vdash t \rhd A}$ t infers the semantic type A in the semantic context Γ

$$\textsc{Var}\ \frac{\Gamma(i) = A}{\Gamma \vdash \mathtt{var}\ i \rhd A} \qquad \textsc{Ascr}\ \frac{\Gamma \vdash A \lhd \qquad B = [\![A]\!]_\Gamma \qquad \Gamma \vdash t \lhd B}{\Gamma \vdash (t : A) \rhd B}$$

$$\textsc{If}\ \frac{\begin{array}{c}\Gamma \vdash b \lhd_v \mathbb{B} \\ \forall(\Xi \rightsquigarrow v) \in [\![b]\!]_\Gamma, \quad (v = \mathtt{tt} \implies \Gamma, \Xi \vdash t \rhd A) \wedge (v = \mathtt{ff} \implies \Gamma, \Xi \vdash u \rhd B)\end{array}}{\Gamma \vdash \mathtt{if}(b, t, u) \rhd \mathtt{let}^{\mathcal{C}}\ v = [\![[\![b]\!]_\Gamma]\!]_\Gamma\ \mathtt{in}\ \mathtt{if}\ v\ \mathtt{then}\ A\ \mathtt{else}\ B}$$

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